$$+\frac{\|A\|\,r}{\Gamma(\alpha-\beta-2)}\int_0^{t_1}\left[(t_2-s)^{\alpha-\beta-2}-(t_1-s)^{\alpha-\beta-2}\right]ds$$

$$+\frac{M}{\Gamma(\alpha-1)}\left[\int_{t_1}^{t_2}(t_2-s)^{\alpha-2}\,ds\right.$$

$$\left.+\int_0^{t_1}\left[(t_2-s)^{\alpha-2}-(t_1-s)^{\alpha-2}\right]ds\right]$$

Again, it is seen that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$. Thus, $\|(Nu)(t_2)-(Nu)(t_1)\| \to 0$, as $t_2 \to t_1$. This shows that the operator N is completely continuous, by the Ascoli-Arzela theorem. Thus, the operator N satisfies all the conditions of Theorem **??**, and hence by its conclusion, either condition (i) or condition (ii) holds. We show that the condition (ii) is not possible.

Let $U=\left\{u\in C^1(J,\mathbb{R}^n) : \|u\|<M\right\}$ with $\max\{l_1,\,l_2\}=l<M$. In view of condition $l<M$ and by (**??**), we have

$$\|Nu\| \le max\{l_1,\,l_2\} < M.$$

Now, suppose there exists $u\in\partial U$ and $\lambda\in(0,1)$ such that $u=\lambda Nu$. Then for such a choice of u and the constant λ, we have

$$M=\|u\|=\lambda\,\|Nu\|<\max\{l_1,l_2\}<M,$$

which is a contradiction. Consequently, by the Leray-Schauder alternative, we deduce that F has a fixed point $u\in\overline{U}$ which is a solution of the boundary value problem (**??**) − (**??**). The proof is completed.

□

We construct an example to illustrate the applicability of the results presented.

Example IV.1. *Consider the following fractional differential equation, for* $t\in J=[0,1]$

$$^cD_{0+}^{\alpha}u(t)-A\,^cD_{0+}^{\beta}u(t)=f\left(t,u(t),^cD_{0+}^{\beta}u(t),^cD_{0+}^{\alpha}u(t)\right), \tag{IV.12}$$

subject to the three-point boundary conditions

$$\begin{cases} u(0)-u(1)-u\left(\frac{1}{2}\right)=1, \\ u'(0)-u'(1)-u'\left(\frac{1}{2}\right)=1, \end{cases} \tag{IV.13}$$

where $\alpha=2,\ \beta=1,\ \lambda=\mu=d=l=1,\ A=\begin{pmatrix}2 & 1\\ 0 & 2\end{pmatrix}$ *and*

$$f_i(t,u,v,w)=\frac{c_i t}{8}arctan(|u|+|v|+|w|),\ i=1,\ 2,$$

such that $f=(f_1,f_2)$ *with* $0<c_i<1,\ i=1,\ 2.$
For every $u_i,\ v_i\in\mathbb{R}^2,\ i=1,\ 2,\ 3,$ *we have*

$$|f_i(t,u_1,u_2,u_3)-f_i(t,v_1,v_2,v_3)| \le \frac{c_i}{8}(|u_1-v_1|$$

$$+|u_2-v_2|+|u_3-v_3|),\ i=1,\ 2,$$

where $L_1=L_2=L_3=\frac{c_i}{8}$ *for appropriate choice of constants* $c_i,\ i=1,2.$ *we check the condition of Theorem* **??**. *Clearly, assumption* (H_1) *holds. A simple computations of* $R_1,\ R_2,\ l_1$ *and* l_2 *shows tha the second condition of Theorems* **??** *and* **??** *is satisfied. Thus the conclusion of Theorems* **??** *and* **??** *applies, and hence the problem* (**??**) − (**??**) *has a unique solution and at least one solution on* $[0,1]$.

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